On Domains of Universal Machines

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Main References

1. Calude, C.S. and Staiger, L.,
   *On universal computably enumerable prefix codes*,

2. Calude, C.S., Nies, A., Staiger, L. and Stephan, F.,
   *Universal Recursively Enumerable Sets of Strings*,
Outline

1. Prefix Codes
2. Description complexity
3. Universal codes
4. Density
5. Spectral properties
6. Plain machines
Fix an alphabet $X = \{0, \ldots, r - 1\}$, $r \geq 2$, and denote by $X^*$ the set of finite strings (words) on $X$.

**Definition (Prefix Code)**

*A subset $V$ of $X^*$ is called a *prefix code* provided $w \sqsubseteq v$ and $w, v \in V$ imply $w = v$.*
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**Definition (Prefix Code)**

A subset \( V \subseteq X^* \) is called a **prefix code** provided \( w \sqsubseteq v \) and \( w, v \in V \) imply \( w = v \).

**Proposition (KRAFT-MCMILLAN inequality)**

Every (prefix) code \( V \subseteq X^* \) satisfies \( \sum_{w \in V} r^{-|w|} \leq 1 \).
Prefix Maximality

Definition (Prefix Maximality)

\[ V \subseteq X^* \text{ is a } \textbf{prefix maximal} \text{ code provided } V \text{ is a prefix code and for every prefix code } W \subseteq X^*, \ V \subseteq W \text{ implies } W = V. \]
Prefix Maximality

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A code $V \subseteq X^*$ is a **prefix maximal** code provided $V$ is a prefix code and for every prefix code $W \subseteq X^*$, $V \subseteq W$ implies $W = V$.

**Fact**

A prefix code $V \subseteq X^*$ is prefix maximal if and only if for every $u \in X^*$ there is a $w \in V$ such that $u \sqsubseteq w$ or $w \sqsubseteq u$. 
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A prefix code $V \subseteq X^*$ is prefix maximal if and only if for every $u \in X^*$ there is a $w \in V$ such that $u \sqsubseteq w$ or $w \sqsubseteq u$.

Fact (Sufficient condition)

A prefix code $V$ is prefix maximal if $\sum_{w \in V} r^{-|w|} = 1$. 
KRAFT’s construction

Theorem (KRAFT’s construction)

Let \( s : \mathbb{N} \rightarrow \mathbb{N} \) be a function such that \( s(0) = 0 \) and \( \sum_{n \in \mathbb{N}} s(n) \cdot r^{-n} \leq 1 \). Then there is a prefix code \( V \subseteq X^* \) such that

\[
\left| \{ w : w \in V \land |w| = n \} \right| = s(n).
\]
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Observe that $\sum_{n \in \mathbb{N}} s(n) = |V|$. 
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Observe that $\sum_{n \in \mathbb{N}} s(n) = |V|$.

Theorem (Maximality of infinite prefix codes)

Let $s : \mathbb{N} \to \mathbb{N}$ be a function such that $\sum_{n \in \mathbb{N}} s(n) \cdot r^{-n} \leq 1$ and
$\sum_{n \in \mathbb{N}} s(n) = \infty$. Then there is a prefix maximal code $V \subseteq X^*$ such that

$$\left| \left\{ w : w \in V \land |w| = n \right\} \right| = s(n).$$
Theorem (Kraft-Chaitin)

Let \( s : \mathbb{N} \to \mathbb{N} \) be a left computable (approximable from below) function such that \( s(0) = 0 \) and \( \sum_{n \in \mathbb{N}} s(n) \cdot r^{-n} \leq 1 \). Then there is a computably enumerable prefix code \( V \subseteq X^* \) such that

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Theorem (Kraft-Chaitin)

Let $s : \mathbb{N} \rightarrow \mathbb{N}$ be a left computable (approximable from below) function such that $s(0) = 0$ and $\sum_{n \in \mathbb{N}} s(n) \cdot r^{-n} \leq 1$. Then there is a computably enumerable prefix code $V \subseteq X^*$ such that

$$\left| \{ w : w \in V \land |w| = n \} \right| = s(n).$$

Corollary

If, moreover, $s : \mathbb{N} \rightarrow \mathbb{N}$ is a computable function then $V$ is also computable.
**Definition (Description complexity $K_\varphi$)**

Let $\varphi : \subseteq X^* \to X^*$ be a partial computable function.

$$K_\varphi(w) := \inf\{|\pi| : \varphi(\pi) = w\}$$

**Definition (Plain or Simple universal machine)**

A machine (mapping) $\mathcal{U}_S : \subseteq X^* \to X^*$ is called **universal** if and only if for every partial computable mapping $\varphi : \subseteq X^* \to X^*$ there is a constant $c_\varphi$ such that

$$\forall w (K_\varphi(w) \leq K_{\mathcal{U}_S}(w) + c_\varphi).$$
### Description complexity: plain complexity

#### Definition (Description complexity $K_\varphi$)

Let $\varphi : \subseteq X^* \rightarrow X^*$ be a partial computable function.

$$K_\varphi(w) := \inf\{|\pi| : \varphi(\pi) = w\}$$

#### Definition (Plain or Simple universal machine)

A machine (mapping) $U_S : \subseteq X^* \rightarrow X^*$ is called universal if and only if for every partial computable mapping $\varphi : \subseteq X^* \rightarrow X^*$ there is a constant $c_\varphi$ such that

$$\forall w (K_\varphi(w) \leq K_{U_S}(w) + c_\varphi).$$

#### Definition (Plain description complexity)

$$C(w) := \min\{|\pi| : U_S(\pi) = w\}$$
Description complexity: prefix-free complexity

Definition (Prefix-free universal machine)

A prefix-free machine (mapping) $\mathcal{U}_P : \subseteq X^* \rightarrow X^*$ is called **universal** if and only if

1. $\text{dom}(\mathcal{U}_P)$ is prefix-free, and

2. for every partial computable mapping $\phi : \subseteq X^* \rightarrow X^*$ with prefix-free domain $\text{dom}(\phi)$ there is a constant $c_\phi$ such that

$$\forall w (K_\phi(w) \leq K_{\mathcal{U}_P}(w) + c_\phi).$$
## Description complexity: prefix-free complexity

### Definition (Prefix-free universal machine)

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\[
\forall w \left( K_\phi(w) \leq K_{\mathcal{U}_P}(w) + c_\phi \right).
\]

### Definition (Prefix-free description complexity)

\[
H(w) := \min \{|\pi| : \mathcal{U}_P(\pi) = w\}
\]
Numbering words

Definition (Quasi-lexicographical order of \(\{0, \ldots, r - 1\}^*\))

\[ w <_{ql} v :\iff |w| < |v| \lor (|w| = |v| \rightarrow 0.w <_{real} 0.v) \]
Numbering words

Definition (Quasi-lexicographical order of \{0, \ldots, r - 1\}^*)

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Not to confuse with the lexicographical order:

\[ \text{e} <_{\text{lex}} 0 <_{\text{lex}} 00 <_{\text{lex}} \cdots <_{\text{lex}} 0^i <_{\text{lex}} \cdots \]

\[ \cdots <_{\text{lex}} 0^l1 <_{\text{lex}} 0^l10 <_{\text{lex}} \cdots <_{\text{lex}} 01 <_{\text{lex}} \cdots <_{\text{lex}} 1 <_{\text{lex}} \cdots \]
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\[ \cdots <_{lex} 0^l 1 <_{lex} 0^l 10 <_{lex} \cdots <_{lex} 01 <_{lex} \cdots <_{lex} 1 <_{lex} \cdots \]

Definition (Complexity of natural numbers)

\[ K_\varphi(n) := K_\varphi(\text{n-th word in } X^* \text{ w.r.t. } \leq_{ql}) \]
**Definition**

*We say that a computably enumerable prefix code $V \subseteq X^*$ is universal if there is a universal prefix-free machine $\emptyset$ such that $V \supseteq \text{dom}(\emptyset)$.*
 Universal c.e. prefix codes

We say that a computably enumerable prefix code $V \subseteq X^*$ is **universal** if there is a universal prefix-free machine $\mathcal{U}$ such that $V \supseteq \text{dom}(\mathcal{U})$.

- Characterise universal c.e. prefix codes.
Universal c.e. prefix codes

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- Are universal c.e. prefix codes computable?
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- How big (set-theoretic, information-theoretic) are universal c.e. prefix codes?
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We say that a computably enumerable prefix code $V \subseteq X^*$ is **universal** if there is a universal prefix-free machine $\mathcal{U}$ such that $V \supseteq \text{dom}(\mathcal{U})$.

- Characterise universal c.e. prefix codes.
- Are universal c.e. prefix codes computable?
- How big (set-theoretic, information-theoretic) are universal c.e. prefix codes?
- Are universal c.e. prefix codes necessarily domains of prefix-free universal machines?
Let $V \subseteq X^*$ be a c.e. prefix code. Then, the following statements are equivalent:

1. The set $V$ is a universal c.e. prefix code.

2. For every c.e. prefix code $D \subseteq X^*$ there exist a partial computable one-one function $\varphi : \subseteq X^* \rightarrow X^*$ and a constant $k \in \mathbb{N}$ such that:
   
   a. $D \subseteq \text{dom}(\varphi)$, $\varphi(D) \subseteq V$, and
   
   b. $|\varphi(u)| \leq |u| + k$, for every $u \in \text{dom}(\varphi)$. 
For the case $V = \text{dom}(\mathcal{U})$, where $\mathcal{U}$ is a prefix-free universal machine we have:

**Corollary**

For every c.e. prefix code $D \subseteq X^*$ and every universal prefix machine $\mathcal{U}$ there are a one-one partial computable function $\varphi : \subseteq X^* \rightarrow X^*$ and a constant $k \in \mathbb{N}$ such that:

a. $D \subseteq \text{dom}(\varphi)$, $\varphi(D) \subseteq \text{dom}(\mathcal{U})$,

b. $|\varphi(u)| \leq |u| + k$, for all $u \in D$, and

c. $\mathcal{U}(\varphi(u)) = u$, for all $u \in D$. 
Computability and set-theoretical maximality

Theorem (Nies, Calude & St.)

No universal c.e. prefix code $V \subseteq X^*$ is computable.
Computability and set-theoretical maximality

Theorem (Nies, Calude & St.)

No universal c.e. prefix code $V \subseteq X^*$ is computable.

Lemma

If $V \subseteq X^*$ is a c.e. prefix maximal code, then $V$ is computable.
Computability and set-theoretical maximality

**Theorem (Nies, Calude & St.)**

*No universal c.e. prefix code $V \subseteq X^*$ is computable.*

**Lemma**

*If $V \subseteq X^*$ is a c.e. prefix maximal code, then $V$ is computable.*

**Corollary**

*No universal c.e. prefix code is a prefix maximal code.*
## Computability and set-theoretical maximality

### Theorem (Nies, Calude & St.)

No universal c.e. prefix code \( V \subseteq X^* \) is computable.

### Lemma

If \( V \subseteq X^* \) is a c.e. prefix maximal code, then \( V \) is computable.

### Corollary

No universal c.e. prefix code is a prefix maximal code.

However:

There are computable prefix codes which are not contained in a computable prefix maximal code.
Definition (Spectrum Function)

Let $L \subseteq X^*$. Then

$$s_L(n, c) := \left| \{ w : w \in L \land n \leq |w| \leq n + c \} \right|$$

is referred to as the spectrum function of $L$. 
Spectrum Function

Definition (Spectrum Function)

Let $L \subseteq X^*$. Then

$$s_L(n, c) := \left| \{ w : w \in L \land n \leq |w| \leq n + c \} \right|$$

is referred to as the **spectrum function** of $L$.

Specials cases

- $s_L(n) := s_L(n, 0)$ is also referred to as the **structure function** of the language $L \subseteq X^*$ [Chomsky and Miller ’58].
- $C_L(n) := s_L(0, n)$ is also referred to as the **census function** of the language $L \subseteq X^*$. 
Fact

If \( W \subseteq X^* \) is computably enumerable (computable) then \( s_W \) is a left computable (computable) function, and if \( W \) is computably enumerable and \( s_W \) is a computable function then \( W \) is also computable.
Fact

If $W \subseteq X^*$ is computably enumerable (computable) then $s_W$ is a left computable (computable) function, and if $W$ is computably enumerable and $s_W$ is a computable function then $W$ is also computable.

Fact (Uniform embedding)

If $W, W' \subseteq X^*$ are computably enumerable and if there is a $c \in \mathbb{N}$ such that $s_W(n, c) \leq s_{W'}(n, c)$ for all $n \in \mathbb{N}$ then there is a one-to-one partial computable function $\varphi : W \rightarrow W'$ such that $||\varphi(w)|| - |w| | \leq c$ for all $w \in W$. 
Fact

If \( V \subseteq X^* \) is a (prefix) code then
\[
\lim_{n \to \infty} r^{-n} \cdot s_V(0, n) = 0.
\]

Definition (Entropy)

\[
H_L := \inf \left\{ \alpha : \alpha \geq 0 \land \sum_{w \in L} r^{-\alpha \cdot |w|} < \infty \right\} = \limsup_{n \to \infty} \frac{\log_r (1 + s_L(0, n))}{n}
\]
### Entropy and Logarithmic Density

**Fact**

If $V \subseteq X^*$ is a (prefix) code then

$$\lim_{n \to \infty} r^{-n} \cdot s_V(0, n) = 0.$$ 

**Definition (Entropy)**

$$H_L := \inf \left\{ \alpha : \alpha \geq 0 \land \sum_{w \in L} r^{-\alpha \cdot |w|} < \infty \right\} = \limsup_{n \to \infty} \frac{\log_r (1 + s_L(0, n))}{n}$$

**Definition (Logarithmic Density)**

$$H_L := \liminf_{n \to \infty} \frac{\log_r (1 + s_L(0, n))}{n}$$
Entropy and Logarithmic Density: Properties

**Fact**

\[
0 \leq H_L \leq H_L \leq 1, \\
H_L \leq H_{L'}, \quad \text{if } L \subseteq L', \quad \text{and} \\
H_{L \cup L'} = \max\{H_L, H_{L'}\}
\]
Fact

\[
0 \leq H_L \leq H_L' \leq 1, \\
H_L \leq H_L', \quad \text{if } L \subseteq L', \quad \text{and} \\
H_{L \cup L'} = \max\{H_L, H_L'\}
\]

Lemma

If \( V \subseteq X^* \) is a universal c.e. prefix code then \( \sum_{w \in L} r^{-|w|} < 1 \), \\
\( \sum_{w \in L} r^{-\alpha \cdot |w|} = \infty \) for \( \alpha < 1 \) and \( H_V = 1 \).
Entropy and Logarithmic Density: Properties

**Fact**

\[ 0 \leq H_L \leq H_L \leq 1, \]
\[ H_L \leq H_{L'}, \quad \text{if } L \subseteq L', \quad \text{and} \]
\[ H_{L \cup L'} = \max\{H_L, H_{L'}\} \]

**Lemma**

*If* \( V \subseteq X^* \) *is a universal c.e. prefix code then* \( \sum_{w \in L} r^{-|w|} < 1, \)
\( \sum_{w \in L} r^{-\alpha \cdot |w|} = \infty \) *for* \( \alpha < 1 \) *and* \( H_V = 1. \)

However:

There are also computable prefix codes \( V \subseteq X^* \) which satisfy
\( \sum_{w \in L} r^{-|w|} < 1, \sum_{w \in L} r^{-\alpha \cdot |w|} = \infty \) *for* \( \alpha < 1 \) *and* \( H_V = 1. \)
SOLOVAY’s universal prefix machine $\mathcal{T}$

**Proposition**

Let $\mathcal{T}$ be SOLOVAY’s universal prefix machine. Then there are an $n_0 \in \mathbb{N}$ and a $d \in \mathbb{N}$ such that

$$r^{n-H(n)-d} \leq s_{dom}(\mathcal{T})(n) \leq r^{n-H(n)+d} \text{ for all } n \geq n_0.$$
**Proposition**

Let $\mathcal{T}$ be SOLOVAY’s universal prefix machine. Then there are an $n_0 \in \mathbb{N}$ and a $d \in \mathbb{N}$ such that

$$r^{n-H(n)-d} \leq s_{\text{dom}(\mathcal{T})}(n) \leq r^{n-H(n)+d} \text{ for all } n \geq n_0.$$ 

**Lemma**

Let $W \subseteq X^*$ be computably enumerable and $\sum_{w \in W} r^{-|w|} < \infty$. Then there is a $d \in \mathbb{N}$ such that

$$s_W(0, n) \leq r^{n-H(n)+d} \text{ for all } n \in \mathbb{N}.$$
### Theorem (Universal c.e. prefix codes)

Let $V \subseteq X^*$ be a computably enumerable prefix code. Then $V$ is a universal c.e. prefix code if and only if there are $c, d \in \mathbb{N}$ such that

$$r^{n - H(n) - d} \leq s_V(n, c) \text{ for all } n \in \mathbb{N}.$$
Theorem (Universal c.e. prefix codes)

Let $V \subseteq X^*$ be a computably enumerable prefix code. Then $V$ is a universal c.e. prefix code if and only if there are $c, d \in \mathbb{N}$ such that

$$r^{n-H(n)-d} \leq s_V(n, c) \text{ for all } n \in \mathbb{N}.$$ 

Theorem (Domains of universal prefix-free machines)

Let $W \subseteq X^*$ be a computably enumerable prefix code. Then $W$ is the domain of a universal prefix-free machine $\mathcal{U}$ if and only if there is a constant $c \in \mathbb{N}$ such that

$$H(\langle n, s_W(n, c) \rangle) \geq n \text{ for all } n \in \mathbb{N}.$$
Example

There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.
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There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.

**Construction:** Let $\mathcal{U}$ be a universal prefix-free machine with
\[ \sum_{w \in \text{dom}(U)} r^{-|w|} < r^{-1} \]
and let
\[ s(n) := \begin{cases} 0, & \text{if } s_{\text{dom}(\mathcal{U})}(n) = 0, \\ r^\lceil \log_r s_{\text{dom}(\mathcal{U})}(n) \rceil, & \text{otherwise.} \end{cases} \]
Example

There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.

**Construction:** Let $U$ be a universal prefix-free machine with
$\sum_{w \in \text{dom}(U)} r^{-|w|} < r^{-1}$ and let

$$s(n) := \begin{cases} 0, & \text{if } s_{\text{dom}(U)}(n) = 0, \\ r^{\lceil \log_r s_{\text{dom}(U)}(n) \rceil}, & \text{otherwise.} \end{cases}$$

Then $s : \mathbb{N} \to \mathbb{N}$ is left computable, $s \geq s_{\text{dom}(U)}$ and
$\sum_{n \in \mathbb{N}} s(n) \cdot r^n \leq 1$. 
### Example

There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.

**Construction:** Let $\mathcal{U}$ be a universal prefix-free machine with $\sum_{w \in \text{dom}(\mathcal{U})} r^{-|w|} < r^{-1}$ and let

$$s(n) := \begin{cases} 0, & \text{if } s_{\text{dom}(\mathcal{U})}(n) = 0, \text{ and} \\ r^{\lceil \log_r s_{\text{dom}(\mathcal{U})}(n) \rceil}, & \text{otherwise.} \end{cases}$$

Then $s : \mathbb{N} \rightarrow \mathbb{N}$ is left computable, $s \geq s_{\text{dom}(\mathcal{U})}$ and $\sum_{n \in \mathbb{N}} s(n) \cdot r^n \leq 1$.

Thus there is a (universal) c.e. prefix code $V$ such that $s_V = s$. 

*Note:* The provided example is a theoretical construct and may not be applicable to practical scenarios.
Example

There is a universal c.e. prefix code which is not the domain of a universal prefix-free machine.

**Construction:** Let $\mathcal{U}$ be a universal prefix-free machine with $\sum_{w \in \text{dom}(U)} r^{-|w|} < r^{-1}$ and let

$$s(n) := \begin{cases} 0, & \text{if } s_{\text{dom}(\mathcal{U})}(n) = 0, \\
 r^\left\lceil \log_r s_{\text{dom}(\mathcal{U})}(n) \right\rceil, & \text{otherwise.}
\end{cases}$$

Then $s : \mathbb{N} \rightarrow \mathbb{N}$ is left computable, $s \geq s_{\text{dom}(\mathcal{U})}$ and $\sum_{n \in \mathbb{N}} s(n) \cdot r^n \leq 1$.

Thus there is a (universal) c.e. prefix code $V$ such that $s_V = s$.

Observe that $s_V(n, c)$ has the form $r^{m_1} + \cdots + r^{m_k}$ with $m_i \leq n + c$ for some $k \leq c + 1$. Hence $H(\langle n, s_V(n, c) \rangle) = O(\log n)$. 
Proposition

If $V \subseteq X^*$ is a universal c.e. prefix code then the number

$$\Omega_V = \sum_{w \in V} r^{-|w|}$$

is a left computable Martin-Löf-random real.
Domains of plain machines

Theorem (Supersets of domains)

Let $W \subseteq X^*$ be computably enumerable. Then $W \supseteq \text{dom}(\mathcal{U})$ for a plain universal machine $\mathcal{U}$ if and only if there is a $c \in \mathbb{N}$ such that

$$r^n \leq s_W(n, c) \text{ for all } n \in \mathbb{N}.$$
Domains of plain machines

**Theorem (Supersets of domains)**

Let $W \subseteq X^*$ be computably enumerable. Then $W \supseteq \text{dom}(\mathcal{U})$ for a plain universal machine $\mathcal{U}$ if and only if there is a $c \in \mathbb{N}$ such that

$$r^n \leq s_W(n, c) \text{ for all } n \in \mathbb{N}.$$ 

**Theorem (Domains of universal plain machines)**

A computably enumerable set $W \subseteq X^*$ is a domain of a universal plain machine if and only if there is a constant $c \in \mathbb{N}$ such that

$$C(s_W(n, c)) \geq n \text{ for all } n \in \mathbb{N}.$$